# STATE SPACE FORMULAS FOR TRANSFER POLES AT INFINITY

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**ABSTRACT**

Formulas for the transfer pole/zero structure at infinity of linear time-invariant systems are given in terms of subspaces of the input and output spaces. The same subspaces are used for characterizing the existence of a transfer function as well as left/right invertibility of a transfer function. For systems that are represented in descriptor form the formulas are given in terms of the matrices $E, A, B, C$ and $D$. No specific assumptions are made on the descriptor representation; in particular, the matrices $E$ and $A$ need not be square.

**Key Words:** Smith-McMillan form, matrix pencils, descriptor representations.

## 1. INTRODUCTION

In this paper we consider transfer (also called transmission) poles and zeros at infinity of a linear time-invariant system. We will give state space expressions for the pole/zero structure at infinity of systems, represented in descriptor form:

$$\begin{aligned} \sigma E\xi &= A\xi + Bu \\ y &= C\xi + Du. \end{aligned} \tag{1.1}$$

Here $\sigma$ denotes differentiation or shift, depending on whether one works in continuous or discrete time. No restrictions will be placed upon the descriptor representation. In particular, it is not required that $sE - A$ is invertible; the matrices $E$ and $A$ may even be nonsquare. It should be noted that, when $sE - A$ is not invertible, a transfer function may still exist. In this paper we will also characterize the existence of a transfer function in terms of the matrices $E$, $A$, $B$, $C$ and $D$. For motivations and proofs the reader is referred to [7] which is an extended version of the present paper.

## 2. POLYNOMIAL CHARACTERIZATION

As a starting point, consider a linear time-invariant system, given by equations of the form

$$R_1(\sigma)y - R_2(\sigma)u = 0. \tag{2.1}$$

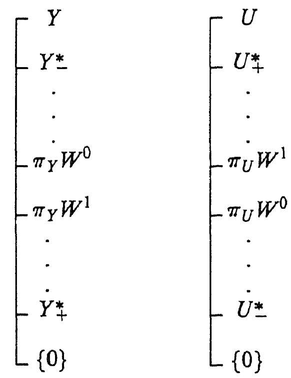
Here $R_1(s)$ is a polynomial matrix of size $r \times p$ and $R_2(s)$ is a polynomial matrix of size $r \times m$. We will call two representations of the form (2.1), given by polynomial matrices $[R_1(s) \quad -R_2(s)]$ and $[\tilde{R}_1(s) \quad -\tilde{R}_2(s)]$, respectively, *input-output equivalent* if the rational vector spaces $\ker[R_1(s) \quad -R_2(s)]$ and $\ker[\tilde{R}_1(s) \quad -\tilde{R}_2(s)]$ are equal. Note that, in case the transfer function of the system exists, input-output equivalence coincides with transfer equivalence (according to which systems are defined to be equivalent if their transfer functions coincide).

Writing the space of outputs and inputs as $W = Y \oplus U$, we define, for each $k \in \mathbb{Z}$, the following sequence of subspaces $W^k \subset W$:

DEFINITION 2.1

$$W^k = \{[y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}} \in W \mid \exists\, [y(s)^{\mathrm{T}} \quad u(s)^{\mathrm{T}}]^{\mathrm{T}} \in W[[s^{-1}]] \text{ such that}$$
$$[s^k R_1(s) \quad -R_2(s)][y(s)^{\mathrm{T}} \quad u(s)^{\mathrm{T}}]^{\mathrm{T}} = 0 \text{ and } [y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}} = [y(\infty)^{\mathrm{T}} \quad u(\infty)^{\mathrm{T}}]^{\mathrm{T}}\}.$$

Here $W[[s^{-1}]]$ denotes the space of proper rational $W$-valued functions. The spaces $W^k$ are obviously invariant under input-output equivalence and we have the following situation:




Here $Y_+^*$ and $U_-^*$ denote the limit spaces of the monotonous sequences $\{\pi_Y W^k\}$ and $\{\pi_U W^{-k}\}$ $(k \geq 0)$, respectively. The spaces $Y_-^*$ and $U_+^*$ are defined analogously.

From Definition 2.1 we can conclude that $\dim Y_+^* = \dim \ker R_1(s)$. This is used in the next theorem. The assumption on $[R_1(s) \quad -R_2(s)]$ that is made in the theorem is of course no restriction under input-output equivalence.

THEOREM 2.2 *Let a system be given by*

$$R_1(\sigma)y - R_2(\sigma)u = 0$$

*where* $[R_1(s) \quad -R_2(s)]$ *is assumed to have full row rank. Then the transfer function* $T(s)$ *of the system exists if and only if the following conditions are satisfied:*

(i) $Y_+^* = \{0\}$

(ii) $U_+^* = U$.

*If these conditions are satisfied, we have*

(iii) $\dim\ker T(s) = \dim U_-^*$

(iv) $\operatorname{rank} T(s) = \dim Y_-^*$.

The next theorem shows that the subspaces $W^k$ determine the pole/zero structure at infinity of the system. The theorem is easily verified by considering the Smith-McMillan form at infinity (see [9]) of the transfer function $T(s)$ (zeros of order $k$ are counted as poles of order $-k$).

THEOREM 2.3 *Let a system be given as in Theorem 2.2. Assume that the transfer function* $T(s) = R_1^{-1}(s)R_2(s)$ *exists. Denote the number of poles at infinity of* $T(s)$ *of order* $\geqslant k$ *by* $p_k$ *and the number of poles at infinity of order* $\leqslant k$ *by* $s_k$ ($k \in \mathbb{Z}$). *Then*

$$p_k = \dim \pi_Y W^k \tag{2.2}$$

*and*

$$s_k = \dim \pi_U W^k - \dim U_-^*. \tag{2.3}$$

## 3. CHARACTERIZATION IN TERMS OF A DESCRIPTOR REPRESENTATION

A rational vector space of the form $\ker[R_1(s) \quad -R_2(s)]$ where $R_1(s)$ and $R_2(s)$ are polynomial matrices can also be represented in the form

$$\ker[R_1(s) \quad -R_2(s)] = H[\ker(sG - F)] \tag{3.1}$$

where $F$, $G$ and $H$ are constant matrices, see [6]. The right-hand side of (3.1) corresponds to a pencil representation

$$\begin{aligned} \sigma G z &= Fz \\ w &= Hz. \end{aligned} \tag{3.2}$$

Here, $F, G: Z \to X$ and $H: Z \to W$; $Z$ is the space of internal variables and $X$ is the equation space. We will call the representation (3.2) *input-output equivalent* to a representation of the form

$$[R_1(\sigma) \quad -R_2(\sigma)]w = 0 \tag{3.3}$$

if (3.1) holds. Note that there always exists a representation of the form (3.3) that is input-output equivalent to a given pencil representation. Therefore the spaces $W^k$ can also be defined for a system that is represented in pencil form ($F$, $G$, $H$): choose $[R_1(s) \quad R_2(s)]$ such that (3.1) holds and apply Definition 2.1. The next lemma serves as a tool for deriving expressions for the spaces $W^k$, $\pi_Y W^k$ and $\pi_U W^k$ in terms of the pencil matrices themselves.

LEMMA 3.1 *Let a system be given by a pencil representation*

$$\begin{aligned} \sigma G z &= Fz \\ y &= H_y z \\ u &= H_u z. \end{aligned} \tag{3.4}$$

*Let* $k \in \mathbb{Z}$, $y_0 \in Y$ *and* $u_0 \in U$. *Then* $[y_0^{\mathrm{T}} \quad u_0^{\mathrm{T}}]^{\mathrm{T}} \in W^k$ *if and only if there exists a rational vector* $z(s)$ *with Laurent expansion*

$$z(s) = z_{-l}s^l + z_{-l+1}s^{l-1} + \cdots + z_0 + z_1 s^{-1} + \cdots$$

*such that the following conditions hold:*

(i) $(sG - F)z(s) = 0$

(ii) *(for* $k \geqslant 0$*)* $H_y z(s)$ *and* $s^k H_u z(s)$ *are proper,* $y_0 = H_y z_0$, $u_0 = H_u z_k$

(iii) *(for* $k \leqslant 0$*)* $s^{-k} H_y z(s)$ *and* $H_u z(s)$ *are proper,* $y_0 = H_y z_{-k}$, $u_0 = H_u z_0$.

Let us now consider a pencil representation of a specific form, namely

$$G = [E \quad 0], \qquad F = [A \quad B], \qquad H_y = [C \quad D], \qquad H_u = [0 \quad I]. \tag{3.5}$$

We are then dealing with a so-called descriptor representation

$$\sigma E\xi = A\xi + Bu$$
$$y = C\xi + Du. \tag{3.6}$$

Here the matrices $E$ and $A$ are not necessarily square; the domain of the mappings $E$ and $A$ will be denoted by $X_d$ (descriptor space) while the codomain will be denoted by $X_e$ (equation space). From Lemma 3.1 we can now get expressions for the spaces $\pi_Y W^k$ and $\pi_U W^k$ in terms of the matrices $E$, $A$, $B$, $C$ and $D$. Combined with Theorem 2.2 and Theorem 2.3 this yields the main theorem of this section. Before presenting the theorem we first introduce the following iterations:

$$N^0 = \{0\}, \qquad N^{m+1} = E^{-1}A[N^m \cap \ker C] \tag{3.7}$$

$$X^0 = X_d, \qquad X^{m+1} = A^{-1}[EX^m + \operatorname{im} B] \tag{3.8}$$

$$T^0 = N^*,$$
$$T^{m+1} = \{\xi \in X_d \mid \exists \bar{\xi} \in T^m, u \in U \text{ with } E\xi = A\bar{\xi} + Bu \text{ and } C\bar{\xi} + Du = 0\} \tag{3.9}$$

$$\bar{T}^0 = X^*, \qquad \bar{T}^{m+1} = A^{-1}E\bar{T}^m \tag{3.10}$$

$$V^0 = X^*,$$
$$V^{m+1} = \{\xi \in X_d \mid \exists u \in U \text{ with } A\xi + Bu \in EV^m \text{ and } C\xi + Du = 0\} \tag{3.11}$$

$$\bar{V}^0 = N^*, \qquad \bar{V}^{m+1} = E^{-1}A\bar{V}^m. \tag{3.12}$$

THEOREM 3.2 *Let a system be given by a descriptor representation $(E, A, B, C, D)$. Let the spaces $N^*$ and $X^*$ be defined as limit spaces of the iterations (3.7) and (3.8), respectively. Let $T^k$, $\bar{T}^k$, $V^k$ and $\bar{V}^k$ $(k \geqslant 0)$ be defined as above and let $T^*$, $\bar{T}^*$, $V^*$ and $\bar{V}^*$ be the corresponding limit spaces. Then we have*

a) *the system has a transfer function if and only if the following conditions are satisfied:*
   (i) $\bar{T}^* \cap N^* \subset \ker C$
   (ii) $\operatorname{im} B \subset A\bar{V}^* + EX^*$.

*Assume that the transfer function $T(s)$ exists. Then we have*

b) $\dim\ker T(s) = \dim\{u \in U \mid \exists \xi \in N^* \text{ with } A\xi + Bu \in EV^* \text{ and } C\xi + Du = 0\}$

c) $\operatorname{rank} T(s) = \dim\{y \in Y \mid \exists \xi \in T^*, u \in U \text{ with } A\xi + Bu \in EX^* \text{ and } y = C\xi + Du\}$.

*Denote the number of poles at infinity of $T(s)$ of order $\geqslant k$ by $p_k$ and the number of poles at infinity of order $\leqslant k$ by $s_k$ $(k \in \mathbb{Z})$. Then we have*

d)
$$p_k = \begin{cases} \dim C[\bar{T}^k \cap N^*] & (\text{for } k \geqslant 1) \\ \dim\{y \in Y \mid \exists \xi \in T^{-k}, u \in U \text{ with } A\xi + Bu \in EX^* \text{ and } y = C\xi + Du\} & (\text{for } k \leqslant 0) \end{cases}$$

$$s_k = \begin{cases} \dim B^{-1}[A\bar{V}^k + EX^*] - \dim U^*_- & (\text{for } k \geqslant 0) \\ \dim \{u \in U \mid \exists \xi \in N^* \text{ with } A\xi + Bu \in EV^{-k-1} \text{ and } C\xi + Du = 0\} \\ \qquad - \dim U^*_- & (\text{for } k \leqslant -1) \end{cases}$$

*where* $U^*_- = \{u \in U \mid \exists \xi \in N^* \text{ with } A\xi + Bu \in EV^* \text{ and } C\xi + Du = 0\}$.

It should be noted that no requirements of e. g. non-redundancy are made on the descriptor representation in the above Theorem. The formulas are less involved when such requirements are made.

REMARK 3.3 For a standard state space representation, i. e. a descriptor representation with $E = I$, the formulas coincide with known expressions for the zeros at infinity as written down in [4,8].This is worked out in [7].

We shall now mention some corollaries that follow from Theorem 3.2. These are all worked out in detail in [7]. First of all, formulas for the zeros at infinity of a pencil $sE - A$ are derived by simply considering the representation

$$y - (\sigma E - A)u = 0 \tag{3.13}$$

which is input-output equivalent to

$$\begin{aligned} \sigma E\xi &= A\xi + y \\ u &= \xi. \end{aligned} \tag{3.14}$$

Along the same lines left and right invertibility of $sE - A$ is characterized in a geometric way. The formulas correspond to known formulas in the literature, as found in [1,2,4].

Next, the approach in this paper enables a geometric proof of the fact that for descriptor representations that are controllable and observable at infinity the transfer zeros at infinity coincide with the invariant zeros at infinity, as defined from the system pencil, whereas the transfer poles at infinity can be calculated as the zeros at infinity of $sE - A$. Under the assumption that $sE - A$ is invertible, this result was proved in [10] in a completely different way. In our result the invertibility of $sE - A$ is not required.

Finally, it should be noted that a transfer function $T(s)$ that corresponds to a descriptor representation $(E, A, B, C, D)$ can be proper, even if $E$ is singular. As a result of Theorem 3.2 necessary and sufficient conditions in terms of $E$, $A$, $B$, $C$ and $D$ can be given to decide whether $T(s)$ is proper:

COROLLARY 3.4 *Let a system be given by a descriptor representation* $(E, A, B, C, D)$ *for which the transfer function* $T(s)$ *exists. Let the spaces* $N^*$ *and* $X^*$ *be defined as before, i. e. as limit spaces of the iterations (3.7) and (3.8), respectively. Then* $T(s)$ *is proper if and only if*

$$N^* \cap A^{-1}EX^* \subset \ker C. \tag{3.15}$$

The above corollary can be made more specific for instance when we assume that the descriptor representation $(E, A, B, C, D)$ satisfies the following conditions (conditions (ii) and (iii) are necessary conditions for minimality w. r. t. the size of $E$, see [3]):

(i) $E$ and $A$ are square

(ii) $[E \quad B]$ has full row rank

(iii) $[E^T \quad C^T]^T$ has full column rank.

We then have $X^* = X_d$ and $N^* = \ker E$ because of conditions (ii) and (iii). According to the above corollary the transfer function is proper if and only if

$$\ker E \cap A^{-1}[\mathrm{im}\, E] = \{0\}. \tag{3.16}$$

By a suitable choice of coordinates we can rewrite the descriptor representation in the form

$$\sigma\xi_1 = A_{11}\xi_1 + A_{12}\xi_2 + B_1 u \tag{3.17}$$

$$0 = A_{21}\xi_1 + A_{22}\xi_2 + B_2 u \tag{3.18}$$

$$y = C_1\xi_1 + C_2\xi_2 + Du. \tag{3.19}$$

Clearly, (3.16) holds if and only if $A_{22}$ is invertible. When $A_{22}$ is invertible, we can rewrite (3.18) as

$$\xi_2 = -A_{22}^{-1}(A_{21}\xi_1 + B_2 u). \tag{3.20}$$

Substitution of this expression into the equations (3.17) and (3.19) leads indeed to an equivalent standard state space representation. The above corollary tells us that, under the assumptions (i)-(iii), this is the only circumstance under which the representation can be rewritten in standard state space form.

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